

|  |  |
|--|--|
| $1, 1, 1, 1, \dots, 1, \dots$  | $\frac{1}{1-z} = \sum_{N \geq 0} z^N$                                  |
| $0, 1, 2, 3, 4, \dots, N, \dots$   | $\frac{z}{(1-z)^2} = \sum_{N \geq 1} N z^N$                            |
| $0, 0, 1, 3, 6, 10, \dots, \binom{N}{2}, \dots$  | $\frac{z^2}{(1-z)^3} = \sum_{N \geq 2} \binom{N}{2} z^N$               |
| $0, \dots, 0, 1, M+1, \dots, \binom{N}{M}, \dots$  | $\frac{z^M}{(1-z)^{M+1}} = \sum_{N \geq M} \binom{N}{M} z^N$           |
| $1, M, \binom{M}{2}, \dots, \binom{M}{N}, \dots, M, 1$   | $(1+z)^M = \sum_{N \geq 0} \binom{M}{N} z^N$                           |
| $1, M+1, \binom{M+2}{2}, \binom{M+3}{3}, \dots$  | $\frac{1}{(1-z)^{M+1}} = \sum_{N \geq 0} \binom{N+M}{N} z^N$           |
| $1, 0, 1, 0, \dots, 1, 0, \dots$   | $\frac{1}{1-z^2} = \sum_{N \geq 0} z^{2N}$                             |
| $1, c, c^2, c^3, \dots, c^N, \dots$  | $\frac{1}{1-cz} = \sum_{N \geq 0} c^N z^N$                             |
| $1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \dots, \frac{1}{N!}, \dots$                                     | $e^z = \sum_{N \geq 0} \frac{z^N}{N!}$                                 |
| $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{N}, \dots$   | $\ln \frac{1}{1-z} = \sum_{N \geq 1} \frac{z^N}{N}$                    |
| $0, 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \dots, H_N, \dots$  | $\frac{1}{1-z} \ln \frac{1}{1-z} = \sum_{N \geq 1} H_N z^N$            |
| $0, 0, 1, 3\left(\frac{1}{2} + \frac{1}{3}\right), 4\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right), \dots$ | $\frac{z}{(1-z)^2} \ln \frac{1}{1-z} = \sum_{N \geq 0} N(H_N - 1) z^N$ |

**Table 3.1** Elementary ordinary generating functions

|  |  |
|--|--|
| $A(z) = \sum_{n \geq 0} a_n z^n$   | $a_0, a_1, a_2, \dots, a_n, \dots$   |
| $B(z) = \sum_{n \geq 0} b_n z^n$   | $b_0, b_1, b_2, \dots, b_n, \dots$   |
| right shift  |  |
| $zA(z) = \sum_{n \geq 1} a_{n-1} z^n$  | $0, a_0, a_1, a_2, \dots, a_{n-1}, \dots$                                      |
| left shift   |  |
| $\frac{A(z) - a_0}{z} = \sum_{n \geq 0} a_{n+1} z^n$                               | $a_1, a_2, a_3, \dots, a_{n+1}, \dots$   |
| index multiply (differentiation)   |  |
| $A'(z) = \sum_{n \geq 0} (n+1)a_{n+1} z^n$   | $a_1, 2a_2, \dots, (n+1)a_{n+1}, \dots$  |
| index divide (integration)   |  |
| $\int_0^z A(t) dt = \sum_{n \geq 1} \frac{a_{n-1}}{n} z^n$                         | $0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \dots, \frac{a_{n-1}}{n}, \dots$        |
| scaling  |  |
| $A(\lambda z) = \sum_{n \geq 0} \lambda^n a_n z^n$                                 | $a_0, \lambda a_1, \lambda^2 a_2, \dots, \lambda^n a_n, \dots$                 |
| addition   |  |
| $A(z) + B(z) = \sum_{n \geq 0} (a_n + b_n) z^n$                                    | $a_0 + b_0, \dots, a_n + b_n, \dots$   |
| difference   |  |
| $(1-z)A(z) = a_0 + \sum_{n \geq 1} (a_n - a_{n-1}) z^n$                            | $a_0, a_1 - a_0, \dots, a_n - a_{n-1}, \dots$                                  |
| convolution  |  |
| $A(z)B(z) = \sum_{n \geq 0} \left( \sum_{0 \leq k \leq n} a_k b_{n-k} \right) z^n$ | $a_0 b_0, a_1 b_0 + a_0 b_1, \dots, \sum_{0 \leq k \leq n} a_k b_{n-k}, \dots$ |
| partial sum  |  |
| $\frac{A(z)}{1-z} = \sum_{n \geq 0} \left( \sum_{0 \leq k \leq n} a_k \right) z^n$ | $a_1, a_1 + a_2, \dots, \sum_{0 \leq k \leq n} a_k, \dots$                     |

**Table 3.2** Operations on ordinary generating functions