Semidefinite relaxation and Branch-and-Bound Algorithm for LPECs

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ABSTRACT

In this paper, we propose branch-and-bound (B&B) algorithm for the global resolution of linear programs with linear complementarities constraints (LPECs). The procedure was motivated by the B&B algorithm proposed by Bard and Moore for linear/quadratic bilevel programs, where complementarities are recursively enforced. Burer and Vandenbusshe recently proposed a semidefinite (SDP) relaxation for the nonconvex quadratic programming problem. We extend the Burer-Vandenbusshe approach LPECs, and propose the use of SDP relaxations to generate bounds at the nodes of the B&B tree. Computational results compare the quality of the bounds given by the SDP relaxation with the ones used by Bard and Moore.

KEYWORDS. Semidefinite relaxation. LPEC. Bilevel program.

PM - Programação Matemática

1 Introduction

Linear programs with complementarity constraints (LPECs) are disjunctive linear optimization problems that contain a set of complementarity conditions. LPECs form a subclass of mathematical programs with equilibrium constraints (MPECs) and include bilevel linear/quadratic programs as a particular case.

The main difficulty on the solution of LPECs is associated to the complementarity constraints, which introduce nonconvexities to the problem and are also responsible for the lack of regularity of feasible points. Since an LPEC is a nonlinear programming problem, it would be natural to apply NLP algorithms for solving it. However, as discussed by Andreani and Martínez in [Andreani (2001)], due to the lack of regularity, these algorithms may have a poor performance on the solution of MPECs or may even be inapplicable. Moreover, as shown by Fampa et. al. in [Fampa (2008)], with an application of bilevel programming in energy markets, NLP algorithms may converge to a local optimum of the problem.

In this paper we propose a Branch and Bound (B&B) algorithm to solve an LPEC, which considers a semidefinite relaxation to the problem in order to compute an upper bound at each node of the enumeration tree. The procedure was motivated by the B&B algorithm proposed by Bard and Moore for the linear/quadratic bilevel problem (LQBP) in 1990, in [Bard (1990)], where the authors reformulate the LQBP as an LPEC and use a B&B scheme to recursively enforce the complementarity constraints.

It is well known that one of the key elements in the construction of a B&B algorithm is the method used to obtain a bound for the subproblem at each node of the B&B tree. These bounds have been obtained by the solution of linear and Lagrangean relaxations, and, more recently, by the solution of semidefinite programming (SDP) relaxations. The development of interior point methods for SDP in the late 1980s [Nesterov (1994)] made it possible to efficiently solve these relaxations and motivated the research in this area. The basic idea on the development of SDP relaxations was introduced by Lovasz e Schirijver in [Lovász (1991)], where they present a reformulation for 0-1 linear integer program, using SDP. Extensions of the Lovasz-Schirijver approach to combinatorial optimization problems are considered in the surveys published by Goemans [Goemans (1997)] and by Helmberg [Helmberg (2002)] and references therein. In [Kojima (2000)], Kojima and Tunçel shows how to extend the approach to any optimization problem that can be expressed by a quadratic objective and quadratic constraints.

Burer e Vandenbussche [Burer (2005)] proposed a B&B algorithm for the nonconvex quadratic programming problem, which is based on the the solution SDP relaxations at each node of the enumeration tree. The authors present SDP relaxations of the Karush-Kuhn-Tucker (KKT) conditions of the quadratic program, based on the SDP relaxations of the integrality constraints of binary variables proposed in [Lovász (1991)]. The numerical results presented in [Burer (2005)] show the strength of the use of SDP relaxations on the solution of global optimization problems.

We extend the Burer-Vandenbusshe approach to the LPEC, and propose the use of SDP relaxations to generate upper bounds for the subproblems considered at the nodes of the enumeration tree of the B&B algorithm presented by Bard and Moore. Computational results compare the quality of the bounds given by the SDP relaxation with the ones given by the linear relaxation used by Bard and Moore. This linear relaxation is obtained when the complementarity constraints on the reformulation of the LQBP as an LPEC, are omitted.

2 The LPEC

We consider in this paper the following LPEC

$$\begin{array}{lll} \text{maximize} & c^1x + d^1y & \\ \text{subject to} & & \\ & A^1x - b^1 \leq 0 & (a) \\ & x \geq 0 & (b) \\ & d^{2T} + Q^{1T}x + Q^{2T}y - B^{2T}\mu + \lambda = 0 & (c) \\ & A^2x + B^2y - b^2 \leq 0 & (d) \\ & (A^2x + B^2y - b^2) \circ \mu = 0 & (e) \\ & y \circ \lambda = 0 & (f) \\ & y, \lambda, \mu \geq 0, & (g) \end{array} \tag{1}$$

where $A^1 \in I\!R^{m_1 \times n_1}$, $A^2 \in I\!R^{m_2 \times n_1}$, $B^2 \in I\!R^{m_2 \times n_2}$, $Q^1 \in I\!R^{n_1 \times n_2}$, $Q^2 \in S_-^{n_2}$, i.e., $Q^2 \in I\!R^{n_2 \times n_2}$ and is symmetric, negative semidefinite and b^1, b^2, c^1, d^1 and d^2 are vectors of conformal dimension.

The constraints $(1a) - (1b)$ are called ordinary and the constraints $(1c) - (1g)$ are called equilibrium constraints. Considering that $S(x,y) = d^{2T} + Q^{1T}x + Q^{2T}y$ is the gradient with respect to y of the quadratic function $s(x,y) = d^2y + x^TQ^1y + \frac{1}{2}y^TQ^2y$, then the equilibrium constraints can be identified as the KKT conditions of

$$\begin{array}{ll} \text{maximize}_y & d^2y + x^TQ^1y + \frac{1}{2}y^TQ^2y \\ \text{subject to} & A^2x + B^2y - b^2 \leq 0 \\ & y \geq 0. \end{array} \tag{2}$$

If we substitute in (1), the equilibrium constraints $(1c) - (1g)$ by (2), we obtain the linear/quadratic bilevel program (LQBP)

$$\begin{array}{llll} \text{maximize}_x & c^1x + d^1y & & \\ \text{subject to} & A^1x \leq b^1, \;\; x \geq 0 & & \\ & \text{maximize}_y & d^2y + x^TQ^1y + \frac{1}{2}y^TQ^2y & \\ & \text{subject to} & A^2x + B^2y \leq b^2, \;\; y \geq 0, & \end{array} \tag{3}$$

where (2) is known as the follower problem.

Let us consider $P := \{(x,y) \geq 0 : A^1x - b^1 \leq 0, A^2x + B^2y - b^2 \leq 0\}$ as the constraint set region corresponding to the bilevel program (3) and $M(x) := \{y : y = \text{argmax}\{d^2y + x^TQ^1y + \frac{1}{2}y^TQ^2y : A^2x + B^2y - b^2 \geq 0, y \geq 0\}\}$ as the reaction set, which contains the reactions of the follower problem in (3) for a given x. We make the following assumptions: P is nonempty and compact and $M(x)$ is a point-to-point map. These assumptions guarantee the existence of a global optimal solution for problem (1).

3 A Semidefinite Programming Relaxation for the LPEC

In this section we extend the SDP relaxation presented by Burer and Vandenbussche for the nonconvex quadratic problem to the LPEC (1). Let's first consider

$$\begin{array}{lcl} G_{xy} & := & \{(\lambda,\mu) \geq 0 : d^{2T} + Q^{1T}x + Q^{2T}y - B^{2T}\mu + \lambda = 0\}, \\ C_{xy} & := & \{(\lambda,\mu) \geq 0 : (A^2x + B^2y - b^2) \circ \mu = 0, y \circ \lambda = 0\}. \end{array}$$

Then problem (1) may be expressed as

$$\begin{array}{ll} \text{maximize} & c^1x + d^1y \\ \text{subject to} & (x,y) \in P \\ & (\lambda,\mu) \in G_{xy} \cap C_{xy}. \end{array} \tag{4}$$

Now, let

$$Z = \begin{pmatrix} 1 \\ x \\ y \\ \lambda \\ \mu \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ \lambda \\ \mu \end{pmatrix}^T = \begin{pmatrix} 1 & x^T & y^T & \lambda^T & \mu^T \\ x & xx^T & xy^T & x\lambda^T & x\mu^T \\ y & yx^T & yy^T & y\lambda^T & y\mu^T \\ \lambda & \lambda x^T & \lambda y^T & \lambda\lambda^T & \lambda\mu^T \\ \mu & \mu x^T & \mu y^T & \mu\lambda^T & \mu\mu^T \end{pmatrix}. \tag{5}$$

The matrix Z is symmetric and positive semidefinite, i.e., $Z \in \mathcal{S}_+^{1+n}$, where $n := n_1 + 2n_2 + m_2$.

If we multiply the constraints $A^1x - b^1 \leq 0$ and $A^2x + B^2y - b^2 \leq 0$ of P and the constraints $d^{2T} + Q^{1T}x + Q^{2T}y - B^{2T}\mu + \lambda = 0$ of G_{xy} by some $\nu \geq 0$ we obtain the quadratic inequalities $A^1x\nu - b^1\nu \leq 0$ and $A^2x\nu + B^2y\nu - b^2\nu \leq 0$ that are valid for P and $d^{2T}\nu + Q^{1T}x\nu + Q^{2T}y\nu - B^{2T}\mu\nu + \lambda\nu = 0$ that are valid for G_{xy}. Therefore, if we define

$$K := \left\{ (x_0, x, y, \lambda, \mu) \in I\!R_+^{1+n} : \begin{array}{l} A^1x - x_0b^1 \leq 0 \\ A^2x + B^2y - x_0b^2 \leq 0 \\ x_0d^{2T} + Q^{1T}x + Q^{2T}y - B^{2T}\mu + \lambda = 0 \end{array} \right\},$$

then the following set represents a set of valid inequalities for (1): $M := \{Z \succeq 0 : Ze_i \in K, i = 2, \ldots, 1+n\}$. where $e_i \in I\!R^{1+n}$ is the vector with all components equal to zero except the i-th component, which is equal to one.

If we consider $Z_{x\mu}$, $Z_{y\mu}$ and $Z_{y\lambda}$ as the submatrices of Z given by $x\mu^T$, $y\mu^T$ and $y\mu^T$, respectively, then the complementarity constraints $(A^2x + B^2y - b^2) \circ \mu = 0$ and $y \circ \lambda = 0$ in C_{xy} may be written in terms of the matrix Z as $\operatorname{diag}(A^2Z_{x\mu} + B^2Z_{y\mu}) = b^2 \circ \mu$ and $\operatorname{diag}(Z_{y\lambda}) = 0$, respectively.

Finally, considering

$$Q := \frac{1}{2} \begin{pmatrix} 0 & c^1 & d^1 & 0 & 0 \\ {c^1}^T & 0 & 0 & 0 & 0 \\ {d^1}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then the objective function of (1), given by $c^1x + d^1y$, may be written as $Q \bullet Z$.

Therefore, we have the following SDP reformulation of (1):

$$\begin{array}{ll} \text{maximize} & Q \bullet Z \\ \text{subject to} & \\ & Z = \begin{pmatrix} 1 & x^T & y^T & \lambda^T & \mu^T \\ x & xx^T & xy^T & x\lambda^T & x\mu^T \\ y & yx^T & yy^T & y\lambda^T & y\mu^T \\ \lambda & \lambda x^T & \lambda y^T & \lambda\lambda^T & \lambda\mu^T \\ \mu & \mu x^T & \mu y^T & \mu\lambda^T & \mu\mu^T \end{pmatrix} \in M \\ & (x,y) \in P \\ & (\lambda,\mu) \in G_{xy} \\ & \operatorname{diag}(A^2Z_{x\mu} + B^2Z_{y\mu}) = b^2 \circ \mu \\ & \operatorname{diag}(Z_{y\lambda}) = 0. \end{array} \tag{6}$$

If we omit the last n columns of equation (5), we arrive at the following linear SDP relaxation of (1):

$$\begin{array}{ll} \text{maximize} & Q \bullet Z \\ \text{subject to} & \\ & Z \in M \\ & Ze_0 = (1; x; y; \lambda; \mu) \\ & (x, y) \in P \\ & (\lambda, \mu) \in G_{xy} \\ & \text{diag}(A^2 Z_{x\mu} + B^2 Z_{y\mu}) = b^2 \circ \mu \\ & \text{diag}(Z_{y\lambda}) = 0. \end{array} \tag{7}$$

Assumption 1 guarantee that (7) has an optimal solution.

4 A Branch-and-Bound Algorithm

We propose the use of a B&B algorithm to solve the LPEC (1). The basic idea of the B&B scheme is to recursively enforce the complementarities through branching.

In order to describe the algorithm, let's consider

$$u := \begin{pmatrix} \mu \\ \lambda \end{pmatrix}, \quad g := \begin{pmatrix} A^2 x + B^2 y - b^2 \\ y \end{pmatrix},$$

and T as the set of indices corresponding to the components of u and g, i.e. $T := \{1, \ldots, m_2 + n_2\}$.

The complementarities in (1) can then be expressed as $u_i g_i = 0$, $\forall i \in T$.

We associate to a particular node l on the B&B tree, two sets T_l^+ and T_l^-, such that $T_l^+ \cup T_l^- \subseteq T$ and $T_l^+ \cap T_l^- = \emptyset$. The constraints $u_j = 0$ and $g_k = 0$ are enforced at node l for every $j \in T_l^+$ and $k \in T_l^-$. If l is the root node, then $T_l^+ = T_l^- = \emptyset$ and if l is a leaf node, $T_l^+ \cup T_l^- = T$. Branching is accomplished at node l by choosing one complementarity constraint $u_{\hat{\imath}} g_{\hat{\imath}} = 0$ to be enforced and creating two children l' and l'', such that

$$\begin{array}{ll} T_{l'}^+ \leftarrow T_l^+ \cup \{\hat{\imath}\}, & T_{l'}^- \leftarrow T_l^-, \\ T_{l''}^- \leftarrow T_l^- \cup \{\hat{\imath}\}, & T_{l''}^+ \leftarrow T_l^+. \end{array}$$

The node l of the tree is, therefore, associated to the subproblem obtained when we add to (1), the following constraints:

$$\begin{array}{ll} u_j = 0, & \forall j \in T_l^+, \\ g_k = 0, & \forall k \in T_l^-. \end{array} \tag{8}$$

In order to obtain an upper bound to this subproblem, we propose the solution of the SDP relaxation derived from (7) where the constraints (8) are added by replacing the sets P, G_{xy} and M, by P_l, G_{xyl} and M_l defined below.

$$P_l := \left\{ (x, y) \geq 0 : \begin{array}{ll} A^1 x - b^1 \leq 0 & \\ A^2 x + B^2 y - b^2 \leq 0 & \\ A_j^2 x + B_j^2 y - b_j^2 = 0, & \forall j \in T_l^{++} \\ y_k = 0, & \forall k \in T_l^{+-} \end{array} \right\},$$

$$
\begin{array}{l}
G_{xyl} := \left\{ (\lambda,\mu) \geq 0: \begin{array}{l} d^{2T} + Q^{1T}x + Q^{2T}y - B^{2T}\mu + \lambda = 0 \\ \mu_j = 0, \quad \forall j \in T_l^{-+} \\ \lambda_k = 0, \quad \forall k \in T_l^{--} \end{array} \right\}, \\
M_l := \{Z \succeq 0 : Ze_i \in K_l, i = 1, \dots, n\}, \\
K_l := \left\{ (x_0, x, y, \lambda, \mu) \in I\!R_+^{1+n} : \begin{array}{ll} A^1x - x_0b^1 \leq 0 & \\ A^2x + B^2y - x_0b^2 \leq 0 & \\ x_0d^{2T} + Q^{1T}x + Q^{2T}y - B^{2T}\mu + \lambda = 0 & \\ A_j^2x + B_j^2y - x_0b_j^2 = 0, & \forall j \in T_l^{++} \\ \mu_j = 0, & \forall j \in T_l^{-+} \\ y_k = 0, & \forall k \in T_l^{+-} \\ \lambda_k = 0, & \forall k \in T_l^{--} \end{array} \right\},
\end{array}
$$

where

$$
\begin{array}{ll}
T_l^{++} := \{i \in T_l^{+} : i \leq m_2\}, & T_l^{+-} := \{i \in T_l^{+} : i > m_2\}, \\
T_l^{-+} := \{i \in T_l^{-} : i \leq m_2\}, & T_l^{--} := \{i \in T_l^{-} : i > m_2\}.
\end{array}
$$

The relaxation solved at node l is therefore given by

$$
\begin{array}{ll}
\text{maximize} & Q \bullet Z \\
\text{subject to} & \\
& Z \in M_l \\
& Ze_0 = (1; x; y; \lambda; \mu) \\
& (x,y) \in P_l \\
& (\lambda,\mu) \in G_{xyl} \\
& \text{diag}(A^2Z_{x\mu} + B^2Z_{y\mu}) = b^2 \circ \mu \\
& \text{diag}(Z_{y\lambda}) = 0,
\end{array}
\tag{9}
$$

5 Numerical Results

In order to test the strength of the SDP relaxation for LPECs, we have implemented the semidefinite-based B&B algorithm, refereed in the section as SDP algorithm, and compared it with the B&B algorithm proposed by Bard and Moore [Bard (1990)], refeered as LP algorithm.

The difference between both algorithms is the method used to compute upper bounds to the subproblems considered at the nodes of the enumeration tree. While our algorithm apply the SDP relaxation (9) to compute the bound, the algorithm presented by Bard and Moore apply the LP relaxation obtained when the complementarity constraints in (1) are omitted and the constraints (8) are added. More specifically, the solution of the following linear program gives the upper bound at node l:

$$
\begin{array}{lll}
\text{maximize} & c^1x + d^1y & \\
\text{subject to} & (x,y) \in P & \\
& (\lambda,\mu) \in G_{xy} & \\
& A_j^2x + B_j^2y - b_j^2 = 0, & \forall j \in T_l^{++} \\
& \mu_j = 0, & \forall j \in T_l^{-+} \\
& y_k = 0, & \forall k \in T_l^{+-} \\
& \lambda_k = 0, & \forall k \in T_l^{--}
\end{array}
\tag{10}
$$

We coded both algorithms in MATLAB 7.2.0.232 and used the toolbox YALMIP R20070810 [Löfberg (2004)] and CSDP 6.0.1 [Borchers 99] to solve the LPs and SDPs relaxations. The experiments were run on a desktop computer with 2.8GHz Intel Celeron and 512 Mb RAM.

The test problems are randomly generated and are instances of the linear bilevel problem represented by 3 with m_1, Q_1 and Q_2 equal to zero and $n_1 + n_2 = 10$. The generator of random problems was also coded in MATLAB and is based on the generator described in [Bard (1990)]. All problems are generated to be bounded, which is guaranteed by the incorporation of the constraint $A_i^2 x + B_i^2 y \leq b_i^2$, in (3), such that all components of A_i^2 and B_i^2 are nonnegative and b_i^2 is equal to sum of the components.

The computational results for the test problems are reported in Figures 1 and 2. In Figure 1, we summarize the results obtained by the solution of 900 random problems. All the problems have four constraints on the follower problem ($m_2 = 4$) and a total of ten variables ($n_1 + n_2 = 10$). The problems were divided into 9 groups, each has a different proportion of the number of variables controlled by the leader (n_1) and by the follower (n_2) of the bilevel program. Note that when increasing the number of variables controlled by the follower, we increase the difficulty of the problem, since the number of complementarities increases, which does not happen if n_1 is increased. We give average results for the 100 problems with each n_2 that were solved by both the LP and SDP algorithms. The vertical axis refers to the average number of nodes on the B&B tree and the the horizontal axis labels n_2. On these problems the strength of the SDP relaxation reduces the the average number of nodes in the B&B tree by a substantial factor that appears to be growing with n_2 for the LP algorithm, and decreasing with n_2 for the SDP algorithm. This result indicates that the SDP relaxation becomes stronger when the proportion of the number of variables controlled by the follower increases, which does not happen with the LP relaxation.

In Figure 2, we summarize the results obtained by the solution of 1000 random problems. All the problems have five variables controled by the leader and five, controled by the follower ($n_1 = n_2 = 5$). The problems were divided into 10 groups, each has a different number of constraints on the follower problem (m_2) which varies from one to ten. We give average results for the 100 problems with each m_2 that were solved by both the LP and SDP algorithms. The vertical axis refers to the average gap between the solution of the relation on the root node of the B&B tree and the optimal solution of the problem, obtained when the B&B algorithms were used. The larger average value for the gap when we use the SDP relaxation is 0,060, for $m_2 = 9$, while the average gaps vary from 28,231 to 308,585 when we use the LP relaxation. The results again confirm the superiority of the SDP relaxation, specially for problems with smaller m_2.

We should remark that the SDP relaxation presented in this paper was motivated by a recent work of Burer and Vandenbussche [Burer (2005)], which provides efficient computational techniques for solving similar SDP problems. Such SDPs are too large to be solved with conventional SDP algorithms as the one used in the current study and, therefore, we were only able to solve small instances of the problem. Our propose in this paper is to show the strongness of the SDP relaxation for LPECs, but to verify its efficiency, we would need to use computational techniques, such as the ones used in [Burer (2005)]. In this paper, the authors show that a similar SDP approach is faster than the LP approach considered to solve large instances of nonconvex quadratic programs.

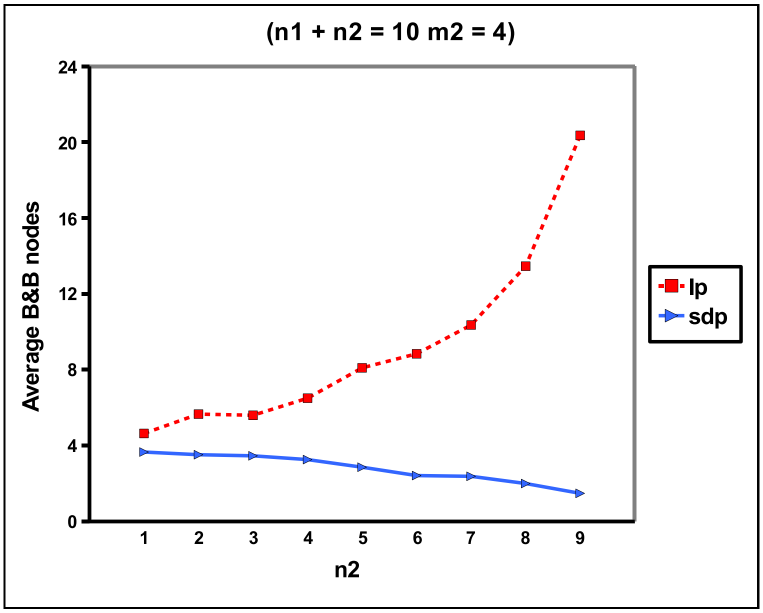

Figure 1: Average number of nodes on the B&B tree

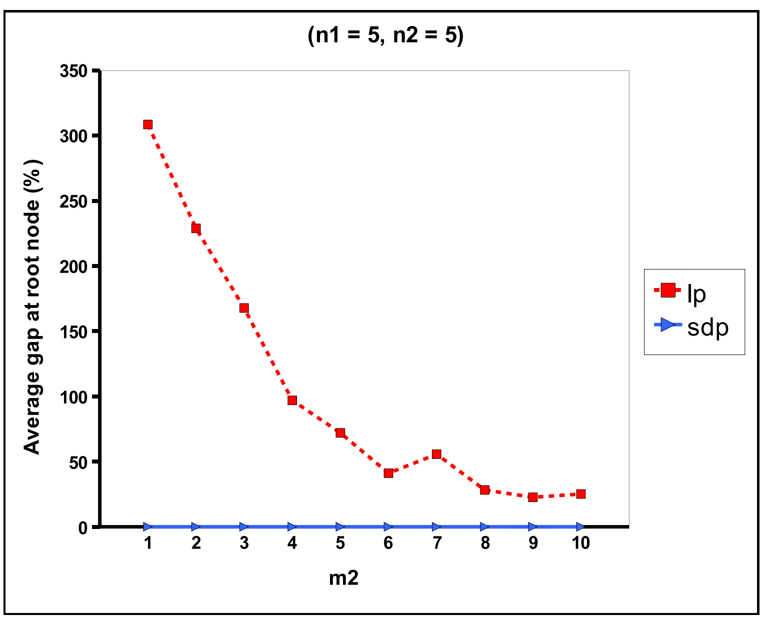

Figure 2: Average gap at the root node of the B&B tree

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